Math 122 / Problem Set 6

Written problems due Monday, October 31

Monday, October 24

1. **(a)** Compute the cosets of the subgroup $H = \{1, x^5\}$ in the dihedral group D_{10} explicitly.

(b) Prove that D_{10}/H is isomorphic to D_5.

(c) Is D_{10} isomorphic to $D_5 \times H$?

2. Prove that the subgroup of $\mathbb{R}^+$ generated by 1 and $\sqrt{2}$ is dense in $\mathbb{R}^+$.

3. Prove that every discrete subgroup of $\mathbf{O}$, the subgroup of the group of motions M that fix the origin, is finite.

Reading: Artin §§5.5, 5.6

Wednesday, October 26

4. Let $\phi : G \to G'$ be a homomorphism, and let S be a set on which G' operates. Show how to define an operation of G on S, using the homomorphism ϕ. If G operates on S, can you define a similar operation of G' on S, using the homomorphism ϕ?

5. $G = D_4$ can be identified with the group of symmetries of a square.

(a) What is the stabilizer of a vertex? An edge?

(b) G acts on the set of two elements consisting of the diagonal lines. What is the stabilizer of a diagonal?

6. Let $G = \mathrm{GL}_n(\mathbb{R})$ operate on the set $S = \mathbb{R}^n$ by left multiplication.

(a) Describe the decomposition of S into orbits for this operation.

(b) What is the stabilizer of the standard basis vector e_1?

7. What is the stabilizer of the coset aH for the operation of G on G/H?

8. A map $S \to S'$ of G-sets is called a *homomorphism* of G-sets if $\phi(gs) = g\phi(s)$ for all $s \in S$ and $g \in G$. Let ϕ be such a homomorphism. Prove the following:

(a) The stabilizer $G_{\phi(s)}$ contains the stabilizer G_s.

(b) The orbit of an element $s \in S$ maps onto the orbit of $\phi(s)$.

Reading: Artin §§5.7, 5.8

Friday, October 28

9. **(a)** Prove that if H and K are subgroups of finite index of a group G, then the intersection $H \cap K$ is also of finite index.

(b) Show by example that the index $[H : H \cap K]$ need not divide $[G : K]$.

10. The purpose of this exercise is to prove Proposition 8.2.

(a) Given an action of G on S, define a map $\phi : G \to \text{Perm}(S)$ by $\phi(g) = m_g$ where m_g is multiplication by g. Show that φ is a homomorphism. Let this map from group actions to homomorphisms be ψ.

(b) Given a homomorphism $\varphi : G \to \text{Perm}(S)$, define an operation of G on S and prove this is a group action. Let this map from homomorphisms to group actions be ψ'.

(c) Show that there is a bijective correspondence between operations of G on S and homomorphisms $G \to \text{Perm}(S)$ by showing that $\psi \circ \psi'$ and $\psi' \circ \psi$ are the identity maps.

Reading: Artin §§6.1, 6.3